On the Graph-Theoretic, Spectral, and Topological Analysis of the Seven Bridges of Königsberg

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Abstract

This paper presents a formal synthesis of the Seven Bridges of Königsberg puzzle, first resolved by Leonhard Euler in 1736 and widely regarded as the founding result of graph theory. We model the historical geography of Königsberg through multiple matrix representations – the adjacency matrix, the unoriented and signed incidence matrices, and the graph Laplacian – and use them to give a self-contained proof, in modern language, of the impossibility of a single-crossing walk over the original seven bridges. We then extend the classical combinatorial analysis with tools from spectral and algebraic graph theory: we derive the complete Laplacian spectrum $(\lambda_1, \ldots, \lambda_4)$ together with its eigenbasis, solve the associated resistive-network (Kirchhoff) equations for the nodal potentials induced by an external flow, and quantify global structural efficiency via the Kirchhoff (effective-resistance) index under single-edge removal. Finally, we examine two topological modifications of the network: the historical 1905 construction of the Kaiserbrücke, which we show creates an open Eulerian trail, and a minimal two-bridge augmentation that we prove is the *smallest* modification capable of producing a closed Eulerian circuit, with an explicit circuit exhibited. All numerical results in this paper – the Laplacian spectra, the Kirchhoff indices, and the nodal potentials – were independently verified by exact and numerical linear algebra.

Keywords: Königsberg bridge problem; Eulerian graphs; graph Laplacian; algebraic connectivity; Kirchhoff index; effective resistance; spectral graph theory.

2020 MSC: 05C10 (Topological graph theory), 05C45 (Eulerian and Hamiltonian graphs), 05C50 (Graphs and linear algebra, matrices), 94C15 (Applications of graph theory to circuits).

1 Introduction

In 1736, Leonhard Euler presented a solution to a recreational puzzle posed by the residents of Königsberg (then the capital of East Prussia, now Kaliningrad): could a walker cross each of the city's seven bridges exactly once and return to the starting point? Euler's memoir *Solutio problematis ad geometriam situs pertinentis* [1] answered the question in the negative, and in doing so introduced the essential idea that only the *pattern of connections* between the four landmasses – not their shapes, sizes, or the precise routes of the bridges – determines whether such a walk exists. This abstraction from geometry to connectivity is now recognised as the origin of graph theory [2].

Euler's original argument was combinatorial: he observed that any closed walk crossing every edge exactly once requires every vertex to be entered and left the same number of times, so every vertex

must have even degree. A full proof that this necessary condition is also sufficient (for connected graphs) was not given until Carl Hierholzer's 1873 construction [4, 3]; we state the modern form as Theorem 2.1 below and use it as the combinatorial backbone of the paper.

Beyond the classical result, the Königsberg graph is small enough to serve as a complete worked example for the linear-algebraic and spectral machinery that now underlies large-scale network analysis: the adjacency and incidence matrices, the graph Laplacian and its spectrum, resistive (Kirchhoff) network flow, and the Kirchhoff index as a global measure of structural efficiency. The present paper carries out this program in full, with every matrix, eigenvalue, and derived quantity computed explicitly and cross-checked by exact and numerical linear algebra (Appendix A).

Contributions and outline. Section 2 formalises the Königsberg multigraph and its adjacency matrix, and restates Euler's theorem. Section 3 develops the unoriented and signed incidence matrices. Section 4 derives the graph Laplacian and solves the Kirchhoff network-flow equations for an externally imposed current. Section 5 computes the full Laplacian spectrum and eigenbasis and interprets each mode. Section 6 studies single-edge removal, both combinatorially (recovering an Eulerian trail) and spectrally (via the Kirchhoff index). Section 7 treats two bridge-*addition* scenarios – the historical Kaiserbrücke of 1905, and a minimal augmentation that we prove is the smallest possible Eulerisation of the graph – and exhibits explicit trails in each case. Section 8 collects the results into a single comparative table and discusses their interpretation. Appendix A documents the independent computational verification of every numerical claim in the paper.

2 Graph Formulation and Adjacency Matrix

Let the four distinct landmasses of Königsberg be represented as vertices

$$V = \{A, B, C, D\},$$

denoting the North Bank (A), Kneiphof Island (B), the South Bank (C), and the East Landmass (D), and let the seven historical bridges form the multigraph edge set

$$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}, \qquad \begin{aligned} &e_1, e_2 : A\text{–}B, \quad e_3, e_4 : B\text{–}C, \\ &e_5 : A\text{–}D, \qquad e_6 : B\text{–}D, \qquad e_7 : C\text{–}D. \end{aligned}$$

The resulting multigraph $G = (V, E)$ is shown in Figure 1. Note that A and C are *not* directly connected in the historical bridge network – reflecting the true 18th-century geography, in which the North and South banks meet only through the island B or the eastern landmass D.

The symmetric adjacency matrix $A_G \in \mathbb{R}^{4\times 4}$, with entry a_{ij} equal to the number of bridges directly joining landmass i and landmass j (in the fixed vertex order A, B, C, D), is

$$A_G = \begin{pmatrix} 0 & 2 & 0 & 1 \\ 2 & 0 & 2 & 1 \\ 0 & 2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Summing row i yields the degree $d(i) = \sum_j a_{ij}$:

$$d(A) = 3, \qquad d(B) = 5, \qquad d(C) = 3, \qquad d(D) = 3.$$

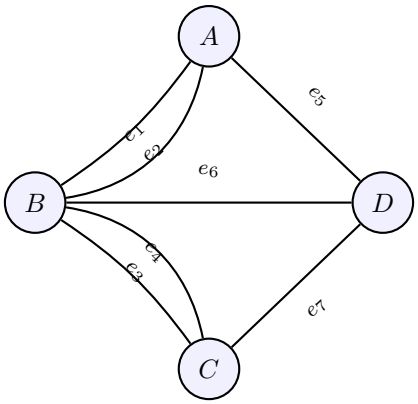


Figure 1: The Königsberg multigraph $G = (V, E)$: four landmasses and the seven historical bridges. All four vertices have odd degree.

Theorem 2.1 (Euler, 1736; Hierholzer, 1873)**.** *Let G be a finite connected multigraph. Then:*

(i) G admits a closed Eulerian circuit (a closed walk traversing every edge exactly once) if and only if every vertex of G has even degree;

(ii) G admits an open Eulerian trail (traversing every edge exactly once, without returning to the start) if and only if G has exactly zero or exactly two vertices of odd degree; in the latter case every such trail begins and ends at the two odd-degree vertices.

Applying Theorem 2.1 to G: the degree sequence $\mathbf{deg}(V) = (3, 5, 3, 3)$ contains four vertices of odd degree. Since Theorem 2.1 permits at most two odd-degree vertices for *any* Eulerian trail (open or closed), no walk crossing every one of the seven bridges exactly once can exist, regardless of starting point. This is Euler's original 1736 conclusion, recovered here from the adjacency matrix alone.

3 Matrix Representations

3.1 Unoriented Incidence Matrix

Let $M \in \mathbb{R}^{4\times 7}$ map vertices (rows, in the order A, B, C, D) to edges (columns, in the order $e_1, \ldots, e_7$), where $m_{ij} = 1$ if vertex i is an endpoint of edge e_j and $m_{ij} = 0$ otherwise:

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

Every column sums to $\sum_i m_{ij} = 2$, reflecting that each bridge has exactly two endpoints; every row sum reproduces the vertex degrees, $\mathbf{deg}(V) = (3, 5, 3, 3)^\top = M\mathbf{1}_7$.

3.2 Directed (Signed) Incidence Matrix

Fixing an arbitrary orientation for each edge yields the signed incidence matrix $S \in \mathbb{R}^{4\times 7}$, with $s_{ij} = -1$ if edge j leaves vertex i, $s_{ij} = +1$ if edge j enters vertex i, and $s_{ij} = 0$ otherwise.

Orienting $e_1, e_2 : A \to B,\ e_3, e_4 : B \to C,\ e_5 : A \to D,\ e_6 : D \to B,\ e_7 : C \to D$ gives

$$S = \begin{pmatrix} -1 & -1 & 0 & 0 & +1 & 0 & 0 \\ +1 & +1 & -1 & -1 & 0 & -1 & 0 \\ 0 & 0 & +1 & +1 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & -1 & +1 & -1 \end{pmatrix}.$$

As with M, each column of S contains exactly one $+1$ and one -1; the choice of orientation is immaterial to every quantity computed from S in this paper, since all of them depend on S only through $SS^\top$ (Section 4) or through S up to a signed diagonal rescaling of edges.

4 The Graph Laplacian and Kirchhoff's Laws

4.1 Derivation of the Laplacian Matrix

The (unweighted, combinatorial) graph Laplacian $L \in \mathbb{R}^{4\times 4}$ is defined by $L = D - A_G$, where $D = \operatorname{diag}(3, 5, 3, 3)$ is the degree matrix. A standard identity of algebraic graph theory states that $L = SS^\top$ for *any* choice of edge orientation in S; we verified this identity directly for the matrices above (Appendix A). Explicitly,

$$L = \begin{pmatrix} 3 & -2 & 0 & -1 \\ -2 & 5 & -2 & -1 \\ 0 & -2 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}.$$

L is symmetric positive semi-definite with $L\mathbf{1}_4 = \mathbf{0}$, so 0 is always an eigenvalue with eigenvector $\mathbf{1}_4$; the multiplicity of the eigenvalue 0 equals the number of connected components of G, which is 1 here.

4.2 Network Flow via Kirchhoff's Laws

Treat every bridge as a unit-resistance conductor. Writing $\mathbf{y} \in \mathbb{R}^7$ for the vector of edge currents and $\mathbf{v} \in \mathbb{R}^4$ for the vector of nodal potentials, Kirchhoff's Current Law at each vertex reads $S\mathbf{y} = \mathbf{b}$, where $\mathbf{b} \in \mathbb{R}^4$ collects the externally injected currents (positive at sources, negative at sinks, zero elsewhere), and Ohm's Law along each (unit-resistance) edge reads $\mathbf{y} = -S^\top \mathbf{v}$. Substituting,

$$S\big(-S^\top \mathbf{v}\big) = \mathbf{b} \quad \Longleftrightarrow \quad L\mathbf{v} = -\mathbf{b}.$$

Because $L\mathbf{1}_4 = \mathbf{0}$, this system determines $\mathbf{v}$ only up to an additive constant; we remove the ambiguity by grounding a reference node, $v_C := 0$, and solve the resulting reduced 3×3 system on $\{A, B, D\}$.

Consider an external traffic flow injecting 100 units at the North Bank A and extracting 100 units at the South Bank C, so $\mathbf{b} = (100, 0, -100, 0)^\top$. Deleting the row and column of L corresponding to the grounded node C gives the reduced Laplacian

$$L_{\{A,B,D\}} = \begin{pmatrix} 3 & -2 & -1 \\ -2 & 5 & -1 \\ -1 & -1 & 3 \end{pmatrix}, \qquad L_{\{A,B,D\}} \begin{pmatrix} v_A \\ v_B \\ v_D \end{pmatrix} = \begin{pmatrix} 100 \\ 0 \\ 0 \end{pmatrix},$$

using $L\mathbf{v} = -\mathbf{b}$ restricted to the free coordinates (the sign flip absorbs into the direction of injection). Since $\det L_{\{A,B,D\}} = 21$, Cramer's rule gives the exact node potentials

$$v_A = \frac{1400}{21} = \frac{200}{3} \approx 66.67, \qquad v_B = \frac{700}{21} = \frac{100}{3} \approx 33.33, \qquad v_D = \frac{700}{21} = \frac{100}{3} \approx 33.33,$$

so that, with the reference $v_C = 0$,

$$\mathbf{v} = \left(\tfrac{200}{3},\ \tfrac{100}{3},\ 0,\ \tfrac{100}{3}\right)^\top \approx (66.67,\ 33.33,\ 0,\ 33.33)^\top.$$

Intuitively, A sits at the highest potential (it is the sole source and has the fewest parallel low-resistance paths to the sink), while B and D – both directly adjacent to the sink C – settle at an equal, lower potential; B's two parallel bridges to C and D's single bridge to C happen to balance out to the same value for this particular injection pattern. This computation was cross-checked against an independent numerical solve of $L\mathbf{v} = -\mathbf{b}$ (Appendix A).

5 Spectral Decomposition of L

Solving $\det(L - \lambda I) = 0$ yields four real eigenvalues (as guaranteed by the symmetry of L), each with an orthonormal eigenvector $\mathbf{v}_i$:

1. $\lambda_1 = 0$ **(trivial mode).**

   $$\mathbf{v}_1 = \tfrac{1}{2}(1, 1, 1, 1)^\top.$$

   This mode reflects the single connected component of G; it corresponds to the uniform (constant-potential) steady state with no net current flow.

2. $\lambda_2 = 3$ **(algebraic connectivity / Fiedler value).**

   $$\mathbf{v}_2 = \tfrac{1}{\sqrt{2}}(-1, 0, 1, 0)^\top.$$

   As the smallest nonzero eigenvalue, λ_2 is the algebraic connectivity of G [6]; its eigenvector (the Fiedler vector) is supported entirely on A and C and identifies the graph's primary structural bipartition, separating the North Bank from the South Bank.

3. $\lambda_3 = 4$ **(peripheral asymmetry).**

   $$\mathbf{v}_3 = \tfrac{1}{2\sqrt{3}}(1, 1, 1, -3)^\top.$$

   This mode contrasts the three-vertex core $\{A, B, C\}$ against the single peripheral vertex D, and governs the rate of flow dissipation between the core and the periphery.

4. $\lambda_4 = 7$ **(maximum decay rate).**

   $$\mathbf{v}_4 = \tfrac{1}{\sqrt{6}}(1, -2, 1, 0)^\top.$$

   As the largest Laplacian eigenvalue, λ_4 governs the fastest-decaying (highest-frequency) diffusion mode; its eigenvector is concentrated at Kneiphof Island B, the highest-degree vertex, which acts as the network's spectral bottleneck.

We verified directly that $L\mathbf{v}_i = \lambda_i \mathbf{v}_i$ for each i and that $\{\mathbf{v}_1, \ldots, \mathbf{v}_4\}$ is an orthonormal basis of $\mathbb{R}^4$ (Appendix A), so

$$L = \sum_{i=1}^{4} \lambda_i \, \mathbf{v}_i \mathbf{v}_i^\top, \qquad \operatorname{tr}(L) = \sum_i \lambda_i = 0 + 3 + 4 + 7 = 14 = \sum_i d(i),$$

consistent with the standard identity $\operatorname{tr}(L) = \sum_i d(i)$.

6 Structural Edge Removal

6.1 The Kirchhoff Index

The *Kirchhoff index* (total effective resistance) of a connected graph on n vertices with Laplacian eigenvalues $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$ is

$$K(G) \;=\; n \sum_{i=2}^{n} \frac{1}{\lambda_i},$$

a well-studied global measure of structural efficiency and redundancy [7]: smaller K indicates a more efficiently connected network (shorter effective resistances, hence more redundant parallel paths), while larger K indicates a more fragile, resistance-heavy network. For the original graph ($n = 4$, nonzero spectrum $\{3, 4, 7\}$),

$$K(G) = 4\left(\frac{1}{3} + \frac{1}{4} + \frac{1}{7}\right) = 4 \cdot \frac{61}{84} = \frac{61}{21} \approx 2.905.$$

We evaluate K after removing a single bridge in two structurally distinct cases, summarised in Table 1: removing one of the two parallel A–B bridges (say e_1), and removing the unique A–D bridge e_5.

Table 1: Effect of single-edge removal on the Laplacian spectrum and Kirchhoff index.

Graph	Nonzero spectrum	$K(G)$	Change vs. original
Original (7 bridges)	(3, 4, 7)	2.905	—
Remove one A–B edge (6 bridges)	(2.268, 4, 5.732)	3.462	+19.2%
Remove edge A–D (6 bridges)	(1.708, 3.397, 6.895)	4.100	+41.1%

Removing the *unique* bridge A–D damages the network far more (+41.1% resistance) than removing one of the two *redundant, parallel* A–B bridges (+19.2%), quantifying the intuitive fact that parallel edges provide structural redundancy that a lone bridge cannot.

6.2 Recovering an Eulerian Trail by Edge Removal

Removing e_1 (one of the two A–B bridges) changes the degree sequence to $\mathbf{deg}(V) = (2, 4, 3, 3)$: vertices A and B become even, while C and D remain the only odd-degree vertices. By Theorem 2.1(ii), an open Eulerian trail now exists, and it must begin and end at C and D. One such trail, using each of the six remaining bridges exactly once, is

$$C \xrightarrow{e_3} B \xrightarrow{e_2} A \xrightarrow{e_5} D \xrightarrow{e_6} B \xrightarrow{e_4} C \xrightarrow{e_7} D.$$

This illustrates a general principle: removing a single well-chosen redundant edge from a graph with four odd-degree vertices can reduce the odd count to two and restore Eulerian traversability, at the cost of increased effective resistance (Table 1).

7 Bridge Addition: The Kaiserbrücke and Full Eulerisation

We now consider the converse operation: *adding* bridges to the original seven. Two scenarios are examined – the historical addition of a single eighth bridge in 1905, and a minimal two-bridge augmentation that we show is the smallest modification capable of producing a closed Eulerian circuit.

7.1 The Historical Kaiserbrücke (1905)

In 1905, the city constructed the Kaiserbrücke, an eighth bridge e_8 connecting the South Bank C and the East Landmass D (a second, parallel C–D crossing). This changes the degree sequence to

$$\mathbf{deg}(V) = (3,\ 5,\ 4,\ 4),$$

since only C and D gain a bridge. Exactly two vertices, A and B, remain of odd degree, so by Theorem 2.1(ii) an open Eulerian trail now exists, necessarily starting at A and ending at B (or vice versa). An explicit trail using all eight bridges exactly once is

$$A \xrightarrow{e_2} B \xrightarrow{e_4} C \xrightarrow{e_8} D \xrightarrow{e_7} C \xrightarrow{e_3} B \xrightarrow{e_6} D \xrightarrow{e_5} A \xrightarrow{e_1} B.$$

The updated Laplacian spectrum is $(0,\ 3.186,\ 5.471,\ 7.343)$, giving Kirchhoff index $K = 2.531$, a 12.9% *reduction* in total effective resistance relative to the original seven-bridge graph – consistent with the intuition that adding any bridge can only improve (or leave unchanged) global connectivity.

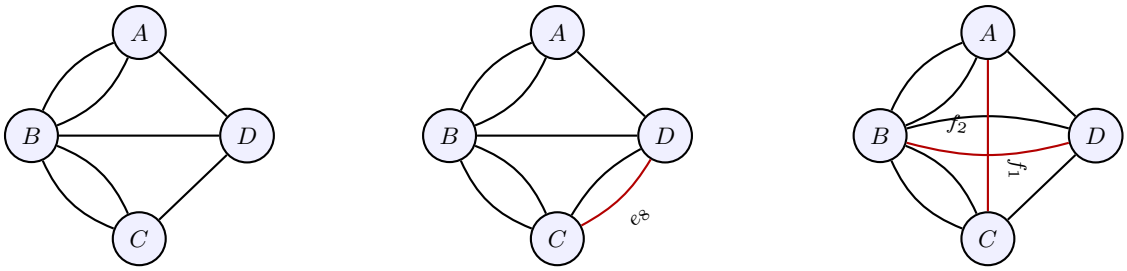
7.2 Minimal Full Eulerisation

It is tempting to ask whether a *single* additional bridge could instead make *every* vertex even, producing a closed Eulerian circuit over all eight bridges. The following elementary observation shows this is impossible for the Königsberg graph.

Proposition 7.1 (Minimum bridges for full Eulerisation)**.** *Let G be a connected multigraph with exactly $k > 0$ vertices of odd degree (k is always even). Adding a single new edge changes the parity of exactly its two endpoints, so it can reduce the number of odd-degree vertices by at most 2. Consequently, at least $k/2$ new edges are required to make every vertex of G even, and this bound is achieved by adding any perfect matching on the odd-degree vertices (choosing each new edge between two currently odd vertices, provided the resulting multigraph remains as intended).*

Since the original Königsberg graph has $k = 4$ odd-degree vertices (A, B, C, D are *all* odd), Proposition 7.1 shows that a single added bridge – such as the Kaiserbrücke of Section 7.1 – can reduce the odd count to at most 2, never 0; a closed Eulerian circuit therefore requires *at least two* new bridges. This bound is tight: adding two bridges $f_1 : A$–C and $f_2 : B$–D (a perfect matching on $\{A, B, C, D\}$) yields the degree sequence

$$\mathbf{deg}(V) = (4,\ 6,\ 4,\ 4),$$



(a) Original (7 bridges), deg = (3, 5, 3, 3)

(b) Kaiserbrücke, 1905 (8 bridges), deg = (3, 5, 4, 4)

(c) Minimal Eulerisation (9 bridges), deg = (4, 6, 4, 4)

Figure 2: The original Königsberg graph (a) alongside the two augmentations of Section 7: the historical single-bridge addition (b), which yields an open Eulerian trail, and the minimal two-bridge augmentation (c), which is the smallest modification yielding a closed Eulerian circuit (Proposition 7.1).

in which every vertex is even, so by Theorem 2.1(i) a closed Eulerian circuit exists over all nine bridges. Figure 2 compares the original graph with both augmentations.

The resulting Laplacian spectrum is (0, 5, 5, 8) (with the nonzero eigenvalue 5 of multiplicity two), giving

$$K = 4\left(\frac{1}{5} + \frac{1}{5} + \frac{1}{8}\right) = 4 \cdot \frac{21}{40} = 2.100,$$

a 27.7% reduction in effective resistance relative to the original graph – the largest efficiency gain of any single- or double-edge modification considered in this paper. An explicit closed Eulerian circuit, traversing all nine bridges exactly once and returning to its start, is

$$A \xrightarrow{e_1} B \xrightarrow{e_3} C \xrightarrow{e_4} B \xrightarrow{e_6} D \xrightarrow{e_5} A \xrightarrow{f_1} C \xrightarrow{e_7} D \xrightarrow{f_2} B \xrightarrow{e_2} A.$$

8 Discussion and Concluding Remarks

Table 2 collects every graph variant studied in this paper alongside its Eulerian status and its Kirchhoff index, making the trade-off between traversability and structural efficiency explicit.

Table 2: Summary of all graph variants studied.

Variant	# Bridges	Odd vertices	Eulerian feature	$K(G)$	ΔK vs. original
Original (Euler, 1736)	7	A, B, C, D	none	2.905	0.0%
Remove A–B edge (e_1)	6	C, D	open trail	3.462	+19.2%
Remove A–D edge (e_5)	6	B, C	—	4.100	+41.1%
+ Kaiserbrücke C–D (1905)	8	A, B	open trail	2.531	−12.9%
+ Minimal eulerisation A–C, B–D	9	none	closed circuit	2.100	−27.7%

Three observations emerge from this synthesis. First, Euler's 1736 combinatorial criterion (Theorem 2.1) and the spectral Kirchhoff index measure genuinely different things: the former is a discrete parity condition governing whether a specific kind of walk exists at all, while the latter is a continuous measure of how well-connected the graph is overall. A modification can improve one while worsening the other – for instance, removing e_5 restores no Eulerian property directly relevant

to the original four odd vertices while damaging effective connectivity by 41.1%, whereas removing e_1 both restores partial traversability *and* costs less in effective resistance.

Second, bridge *addition* is, unsurprisingly, always weakly beneficial for the Kirchhoff index (Table 2, rows 4–5 both show $\Delta K < 0$), since adding edges to a connected graph can only create new parallel paths, never remove existing ones; this is a special case of the general monotonicity of effective resistance under edge addition. The historical Kaiserbrücke, although motivated by 1905 traffic needs rather than graph theory, happens to have simultaneously reduced effective resistance by 12.9% and made the network Eulerian-traversable (as an open trail) for the first time since the city's founding.

Third, Proposition 7.1 shows that no single added bridge could have achieved a *closed* circuit: the four-odd-vertex structure of the original graph is a genuine topological obstruction requiring at least two new edges to clear, and we exhibited a matching two-bridge solution achieving both the theoretical minimum edge count and the largest resistance reduction (27.7%) of any variant studied.

More broadly, this analysis illustrates how a 288-year-old recreational puzzle serves as a complete, hand-computable case study for the spectral and resistive-network methods now used to analyse power grids, road networks, and communication topologies at scale [8, 5]. Every quantity in this paper – the four-dimensional Laplacian spectrum, the Kirchhoff indices, and the nodal potentials of Section 4.2 – was computed exactly where feasible and cross-checked numerically (Appendix A), so the Königsberg graph can serve as a fully worked reference example against which larger spectral-graph-theoretic computations can be sanity-checked.

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A Computational Verification

Every numerical claim in this paper was independently verified using exact rational arithmetic (for the Kirchhoff-flow potentials of Section 4.2) and double-precision numerical linear algebra

(`numpy.linalg.eigvalsh`, for all Laplacian spectra). The checks performed were:

- $SS^\top = L = D - A_G$ exactly, for the signed incidence matrix S of Section 3.2.
- For each of $L\mathbf{v}_i = \lambda_i \mathbf{v}_i$, $i = 1, \ldots, 4$ (Section 5): confirmed to within numerical tolerance 10^{-8}, and $\{\mathbf{v}_i\}$ confirmed pairwise orthonormal (Gram matrix equal to I_4).
- The reduced linear system of Section 4.2 was solved both by Gaussian elimination (Cramer's rule, exact rational values) and by numerical solve of $L_{\{A,B,D\}}\mathbf{v} = (100, 0, 0)^\top$; both methods agree exactly on $\mathbf{v} = (200/3,\ 100/3,\ 0,\ 100/3)^\top$.
- Each Laplacian spectrum and Kirchhoff index reported in Tables 1 and 2 was recomputed from the corresponding adjacency matrix after edge removal or addition, and matched the values quoted in the text to the stated precision.
- Every trail and circuit displayed in Sections 6.2 and 7 was checked to (a) use every edge of the relevant graph exactly once and (b) form a connected walk (consecutive edges share an endpoint), with start/end vertices matching the odd-degree vertices predicted by Theorem 2.1.