Searching for a Decreasing Function in the Collatz Conjecture

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Abstract

This paper investigates whether a potential (Lyapunov-style) function can be found that decreases along Collatz trajectories, and concludes that no such function is established here: the work is a structured research programme, not a proof. The Collatz conjecture asserts that repeated application of the map below takes every positive integer to 1.

$$T(n) = \begin{cases} n/2, & n \text{ even}, \\ 3n+1, & n \text{ odd}. \end{cases}$$

We begin with the obvious candidate $F(n) = n$ and show why it fails. We then move to $F(n) = \log_2 n$, which reveals an average downward drift once the process is compressed to odd integers. Next, residue classes modulo powers of 2 are used to capture how many divisions by 2 follow each odd step. Finally, we formulate a computational experiment that searches for residue-dependent corrections to the logarithmic potential. The aim is to show how an elementary-looking problem becomes a serious investigation spanning dynamical systems, probability, number theory, optimisation and computation. **Keywords:** Collatz conjecture, $3n+1$ problem, Lyapunov function, potential function, 2-adic valuation, modular arithmetic, dynamical systems, computational number theory.

1 The Collatz conjecture as a dynamical system

The conjecture [1, 2] states that for every positive integer n there is some $k \geq 0$ with $T^k(n) = 1$, where T^k means applying T exactly k times. For example, $13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$.

The simplicity is deceptive. A trajectory can climb dramatically before descending: 27 produces a long, wandering trajectory before reaching 1. Proving that many numbers decrease is therefore insufficient; an argument must cover every positive integer.

It helps to view T as a dynamical system on the positive integers. Each starting value n generates an orbit $\{n, T(n), T^2(n), \dots\}$, and the conjecture says every orbit eventually enters the cycle $1 \to 4 \to 2 \to 1$. Hence the conjecture can fail in exactly two ways:

- **Another cycle exists.** There could be a cycle $a_1 \to a_2 \to \cdots \to a_r \to a_1$ that does not contain 1; lower bounds on the length of any such cycle are known [5].
- **An orbit escapes.** A trajectory might never enter a cycle and instead grow without bound.

Any proof must rule out both possibilities.

2 Searching for a quantity that decreases

A standard strategy in mathematics is to find a quantity that decreases. Suppose there were a function F from the positive integers to the reals with $F(T(n)) < F(n)$ for every $n > 1$. Then $F(n) > F(T(n)) > F(T^2(n)) > \cdots$, and if F were bounded below in a suitable sense, infinite descent would be impossible. The central question of this paper is: can we construct such an F?

2.1 Candidate 1: $F(n) = n$

The most obvious candidate fails immediately: $3 \to 10$, so $F(10) > F(3)$. Ordinary size is not monotone along a trajectory, and any successful potential must tolerate temporary increases.

2.2 Candidate 2: logarithmic size

A natural improvement is $F(n) = \log_2 n$, because multiplication becomes addition. An even step $n \to n/2$ changes F by exactly -1:

$$\log_2\left(\tfrac{n}{2}\right) - \log_2 n = -1.$$

An odd step $n \to 3n + 1$ changes F, for large n, by approximately $\log_2 3 \approx 1.585$:

$$\log_2(3n + 1) - \log_2 n \approx \log_2 3.$$

So an even step costs -1 and an odd step costs about $+1.585$. F still does not decrease at every step, but something more interesting appears when steps are grouped.

3 The accelerated map and modular structure

3.1 Compressing to odd integers

If n is odd, $3n + 1$ is even, so we can divide by 2 repeatedly until another odd number appears. Let $v_2(m)$ be the exponent of 2 in the prime factorisation of m (for example $40 = 2^3 \cdot 5$, so $v_2(40) = 3$). The accelerated Collatz map on odd n is

$$U(n) = \frac{3n + 1}{2^{v_2(3n+1)}}.$$

For $n = 13$, $3n + 1 = 40 = 2^3 \cdot 5$, so $U(13) = 5$, and the trajectory becomes $13 \to 5 \to 1$ with all intermediate even numbers removed.

3.2 Logarithmic change under U

Write $k = v_2(3n + 1)$. Then $U(n)/n = (3n + 1)/(2^k n)$, which for large n is approximately $3/2^k$. Taking logarithms, the change in $\log_2 n$ is approximately $\log_2 3 - k$:

k	Approximate change in $\log_2 n$
1	$+0.585$
2	-0.415
3	-1.415
4	-2.415
5	-3.415

Whenever $k \geq 2$ the compressed step decreases logarithmic size; only $k = 1$ produces growth.

3.3 Modular arithmetic explains k

Take n odd and consider n mod 4.

- $n \equiv 1 \pmod 4$. Then $3n + 1 \equiv 3(1) + 1 \equiv 0 \pmod 4$, so $4 \mid 3n + 1$ and $k \geq 2$. The compressed step shrinks n by roughly a factor of at least $3/4$.
- $n \equiv 3 \pmod 4$. Then $3n+1 \equiv 3(3)+1 \equiv 10 \equiv 2 \pmod 4$, so $k = 1$. This is the potentially increasing case.

This is the start of a deeper structure. Examining n mod $8, 16, 32, \ldots$ tells us more precisely how many divisions by 2 occur. For instance, if $3n + 1 \equiv 0 \pmod{16}$ then $k \geq 4$ and the compressed step is approximately $n \to 3n/16$, a substantial decrease.

4 A residue-dependent potential

Instead of treating every integer identically, we assign a correction according to its residue class modulo 2^r:

$$F(n) = \log_2 n + g\big(n \bmod 2^r\big).$$

Here $\log_2 n$ measures size and g supplies a residue-based correction. With $r = 4$ there are only 16 residue classes, so instead of searching over an arbitrary function on infinitely many integers we search for 16 parameters $g(0), g(1), \ldots, g(15)$. This turns part of the problem into an optimisation problem.

4.1 Turning the idea into inequalities

For a transition $n \to T(n)$ we want $F(T(n)) < F(n)$. Substituting and rearranging gives

$$g\big(T(n) \bmod 16\big) - g\big(n \bmod 16\big) < \log_2 \frac{n}{T(n)}.$$

Each Collatz transition therefore yields an inequality in the g-values. An even transition $n \to n/2$ gives $g\big((n/2) \bmod 16\big) - g\big(n \bmod 16\big) < 1$; an odd transition gives another. Thousands or millions of such inequalities can be generated computationally and tested for joint feasibility, for example by linear programming. If the system is infeasible, we learn a concrete limitation of this model; if feasible over a large range, we have an interesting candidate, though still not a proof.

5 What computation can and cannot show

Suppose we test $n = 1, 2, \ldots, 10^8$ and every number reaches 1. This is useful evidence (verification has in fact been pushed far beyond this range [6]), but it establishes only that Collatz holds for the tested range, not for all positive integers. An infinite statement cannot normally be proved by checking finitely many cases, although rigorous partial results exist, such as Tao's theorem that almost all orbits attain almost bounded values [7]. Computation is therefore used here differently: to discover mathematical structure that might eventually support a proof.

5.1 Why the average argument is not enough

Heuristically [3, 4], the average value of $k = v_2(3n+1)$ is about 2, so the expected change per compressed step is

$$\mathbb{E}[\Delta \log_2 n] \approx \log_2 3 - 2 \approx -0.415.$$

This suggests a downward drift. But average behaviour is not the behaviour of every trajectory: an average can be negative while an exceptional trajectory behaves completely differently. The conjecture is precisely about eliminating the possibility of such an exceptional trajectory, so a negative average drift, however suggestive, is not a proof of universal descent.

6 Proposed research experiments

Three experiments structure the investigation.

Experiment	**Question**	**What is measured**
A: ordinary logarithmic potential	Does $F(n) = \log_2 n$ decrease along trajectories? It is already known not to.	The first and most extreme increases.
B: accelerated logarithmic potential	For odd n, with $U(n) = (3n+1)/2^{v_2(3n+1)}$, is $R(n) = U(n)/n$ below or above 1?	How frequently $R(n) < 1$ (decrease) and $R(n) > 1$ (increase) occur.
C: residue-dependent potential	Is there g such that $F(n) = \log_2 n + g(n \bmod 16)$ satisfies $F(U(n)) < F(n)$ for as many n as possible?	Feasibility of g, and where it fails.

If no g exists in Experiment C, we learn something important about the limits of this particular model. If one works over a huge range, we have a candidate but not a proof.

6.1 Richer potentials

A more sophisticated candidate could combine several pieces of arithmetic information:

$$F(n) = \log_2 n + g_1\big(n \bmod 2^r\big) + g_2\big(n \bmod 3^s\big) + \cdots$$

The goal is to encode enough arithmetic structure that temporary increases are compensated by decreases in the correction terms.

6.2 A problem in several fields

The Collatz problem can be viewed as several problems at once: **number theory** ($3n+1$, powers of 2, congruences, valuations); **dynamical systems** (the orbit $n, T(n), T^2(n), \dots$); **probability** (the apparent distribution of $v_2(3n+1)$); **optimisation** (finding a potential satisfying a large family of inequalities); and **computation** (exploring trajectories and testing conjectured structures).

7 What would constitute a proof, and next steps

Finding a candidate F computationally is only the beginning. A proof would need to show mathematically that, for every relevant n, $F(T^k(n)) < F(n)$ for some appropriate k, or establish another rigorous mechanism forcing trajectories downward. It would also have to rule out:

1. nontrivial cycles;
2. trajectories escaping to infinity;
3. exceptional residue patterns;
4. any other mechanism preventing eventual arrival at 1.

7.1 Conclusion

The investigation moves from the naive question "does every number reach 1?" to a sharper one: can we construct a global potential that forces every trajectory toward the 1-cycle? The logarithmic analysis gives a reason to expect a downward tendency, since $\log_2 3 - 2 < 0$. Modular arithmetic explains why that tendency is not uniform. The function $F(n) = \log_2 n + g(n \bmod 2^r)$ is a concrete object to experiment with. The crucial obstacle remains: a downward tendency is not a proof of universal descent.

7.2 Next stage: the computational programme

The natural next step is a program that:

1. generates Collatz trajectories;
2. calculates $v_2(3n+1)$;
3. measures the logarithmic drift;
4. searches residue classes modulo $4, 8, 16, 32, \ldots$;
5. attempts to find suitable g-values;
6. identifies precisely where each candidate potential fails;
7. produces tables and graphs of the results.

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