The Echo-Square Coprime Sequence: A New Integer Sequence with Index-Dependent Arithmetic and Digit-Sum Constraints

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Abstract

We introduce the *Echo-Square Coprime (ESC) sequence*, a new integer sequence generated by a greedy smallest-unused-integer construction subject to three simultaneous constraints binding each term $a(n)$ to its own index n: coprimality with n, a triangular digit sum, and a Boolean "square echo" requiring $a(n)$ to be a perfect square if and only if n is a perfect square. We compute the first 200 terms, prove the square-echo property holds exactly by construction, and characterize a striking emergent consequence: at every perfect-square index the sequence exhibits a sharp upward spike, because integers that are simultaneously a perfect square, coprime to n, and digit-sum-triangular are far sparser than the pool available at ordinary indices. We show the sequence is *not* a permutation of the positive integers — unlike well-known self-avoiding constructions such as the Golomb or EKG sequences — with 226 of the first 300 positive integers absent from the first 200 terms. We verify, via the Online Encyclopedia of Integer Sequences (OEIS) and site-restricted web search, that no published sequence matches this exact term list, and we discuss the methodological limits of that claim honestly. All results are computationally reproducible; the generating algorithm and extended term tables are given in Appendix A.

1 Introduction

Integer sequences defined by a *greedy, self-avoiding rule* — "$a(n)$ is the smallest positive integer not yet used satisfying property P" — have a long recreational and research history [6, 7]. The Golomb self-describing sequence (OEIS A001462) [1], Recamán's sequence (A005132) [2], the EKG/ECG sequence (A064413, built from $\gcd(a(n), a(n-1)) > 1$) [3], and the Yellowstone permutation (A098550) [4] are canonical examples. In nearly all of these, property P relates $a(n)$ only to one or two *preceding terms* of the sequence itself; the index n plays no direct arithmetic role beyond ordering.

The ESC sequence explores a different design space: it ties $a(n)$ directly to its own index n through three heterogeneous number-theoretic predicates operating in different domains — a multiplicative one (gcd-coprimality), a digital one (base-10 digit-sum triangularity), and a self-similar structural one (perfect-squareness of n echoed into perfect-squareness of $a(n)$). No single one of these constraints is unusual on its own; their conjunction, applied index-wise rather than term-wise, is. As we show in Section 5, this conjunction produces a large-scale emergent feature — periodic spikes exactly at square indices — that was not designed in directly but falls out necessarily from the definition.

The remainder of the paper is organized as follows. Section 2 gives the formal definition. Section 3 addresses well-definedness (does $a(n)$ always exist?). Section 4 reports computational

results for the first 200 terms. Section 5 states and proves the square-echo theorem and documents the resulting spike structure. Section 6 shows the sequence fails to be a permutation of $\mathbb{N}$, unlike its predecessors. Section 7 describes the novelty-verification methodology used to check the sequence against OEIS and the wider web. Section 8 discusses open problems, and Section 9 concludes. Appendix A gives the reference algorithm and extended data tables.

2 Formal Definition

Notation. For a positive integer x, let $s(x)$ denote its base-10 digit sum. Let $\mathrm{Sq}(x)$ be true iff x is a perfect square (there exists a nonnegative integer k with $k^2 = x$). Let $T(x)$ be true iff x is a triangular number, i.e. $x = k(k+1)/2$ for some nonnegative integer k — equivalently, $8x + 1$ is a perfect square [8]. The relevant triangular numbers for digit sums are $1, 3, 6, 10, 15, 21, 28, 36$, and 45 (digit sums of ordinary decimal integers range from 1 up to 9 times the number of digits).

Definition 1 (Echo-Square Coprime sequence)**.** *Set* $a(1) = 1$*. For* $n \geq 2$*,*

$$a(n) \;=\; \min\Big\{\, m \in \mathbb{Z}^+ \;:\; m \notin \{a(1), \ldots, a(n-1)\},\ \gcd(m, n) = 1,\ T(s(m)),\ \mathrm{Sq}(m) = \mathrm{Sq}(n) \,\Big\}.$$

In words: $a(n)$ is the smallest positive integer not already used earlier in the sequence such that (i) it is coprime to its own index n, (ii) its digit sum is a triangular number, and (iii) it is a perfect square exactly when n is a perfect square. Condition (iii) is the "echo": squareness of the index is mirrored, term by term, into squareness of the value.

Note that at $n = 1$, $\mathrm{Sq}(1)$ is true ($1 = 1^2$), $\gcd(1, 1) = 1$, and $s(1) = 1$ is triangular, so $a(1) = 1$ is consistent with the general rule and no separate base case is strictly required — we state it explicitly only for clarity.

3 Well-Definedness

Because the definition of $a(n)$ is a greedy minimum over a set carved out by three simultaneous constraints, it is not obvious a priori that this set is nonempty for every n — unlike, say, Recamán's sequence, where the rule is unconditional. We address this in two ways.

Computational evidence. We generated the sequence for $n = 1$ through 200 using an exhaustive linear search over unused candidates m, with a per-step search cap of 5,000,000. The construction succeeded at every step with no failures, and every emitted term was independently re-verified against all three conditions after generation (see Appendix A for the algorithm). No search ever approached the cap; the largest candidate scan needed for any of the first 200 terms stayed several orders of magnitude below it.

Heuristic density argument. For fixed n, write the three constraints as independent-ish filters on the positive integers: (i) coprimality to n has density $\varphi(n)/n > 0$, where φ is Euler's totient function [8]; (ii) triangular digit sum has a fixed positive density among integers of any given digit length, since the triangular numbers up to $9k$ (k = number of digits) form a positive fraction of possible digit sums; (iii) the square-echo condition is the delicate one — when n is *not* a perfect square, "not a perfect square" has density 1 (squares have density 0 among the integers), so this constraint is essentially free; when n *is* a perfect square, the constraint restricts candidates to the perfect squares themselves, which have density $O(1/\sqrt{x})$ — sparse, but still infinite in number.

Treating the three filters as approximately independent, the expected number of valid candidates below X grows like a positive constant times X (non-square n) or like a positive constant times $\sqrt{X}$ (square n) — in both cases unbounded as $X \to \infty$, which is why a solution is expected to exist at every step, and explains why square indices require a substantially larger search (Section 5). We stress this is a heuristic, not a proof: the filters are not truly independent, and a rigorous existence proof for all n remains open (Section 8).

4 Computational Results

The first 30 terms, generated by the algorithm in Appendix A, are given in Table 1.

Table 1: First 30 terms of the ESC sequence.

n	$a(n)$	n	$a(n)$	n	$a(n)$
1	1	11	24	21	73
2	3	12	37	22	69
3	10	13	28	23	60
4	361	14	33	24	91
5	6	15	46	25	2116
6	19	16	1225	26	87
7	12	17	30	27	82
8	15	18	55	28	109
9	64	19	42	29	78
10	21	20	51	30	127

Excluding the perfect-square indices, the ratio $a(n)/n$ over the first 200 terms ranges from 1.200 to 6.228 with a mean of 3.870 — roughly linear growth with substantial local variation. Nineteen of the first 200 terms are prime ($3, 19, 37, 73, 109, 127, 163, 181, 271, 307, 433, 523, 541, 613, 631, 811, 1009, 1063, 1117, \ldots$) showing the sequence is not restricted to any single residue or primality class.

5 Theorem: Square-Index Echo Structure

Proposition 1 (Square Echo)**.** *For every $n \geq 1$, $a(n)$ is a perfect square if and only if n is a perfect square.*

Proof. Immediate from the definition: condition (iii), $\mathrm{Sq}(m) = \mathrm{Sq}(n)$, is imposed as a hard requirement on every candidate m considered in the minimization defining $a(n)$. Any m for which $\mathrm{Sq}(m) \neq \mathrm{Sq}(n)$ is excluded from the candidate set before the minimum is taken, so the selected $a(n)$ satisfies $\mathrm{Sq}(a(n)) = \mathrm{Sq}(n)$ by construction. □

While the proposition is immediate, its *quantitative* consequence is not: Table 2 shows $a(n)$ at every perfect-square index $n \leq 196$, alongside the corresponding integer square roots.

Compare $a(n)$ at these indices to the roughly linear trend ($a(n) \approx 3.9 \times n$) that holds at non-square indices: $a(100) = 26569$ is about 266 times its index, versus a mean ratio of 3.87 elsewhere. The mechanism is a density argument. At a non-square index n, the candidate pool is drawn from *all* non-square integers, which have density 1 (squares have density zero), so the pool essentially coincides with the ordinary integers and the search resolves close to the previous frontier of used numbers. At a square index n, the candidate pool collapses to the perfect squares alone

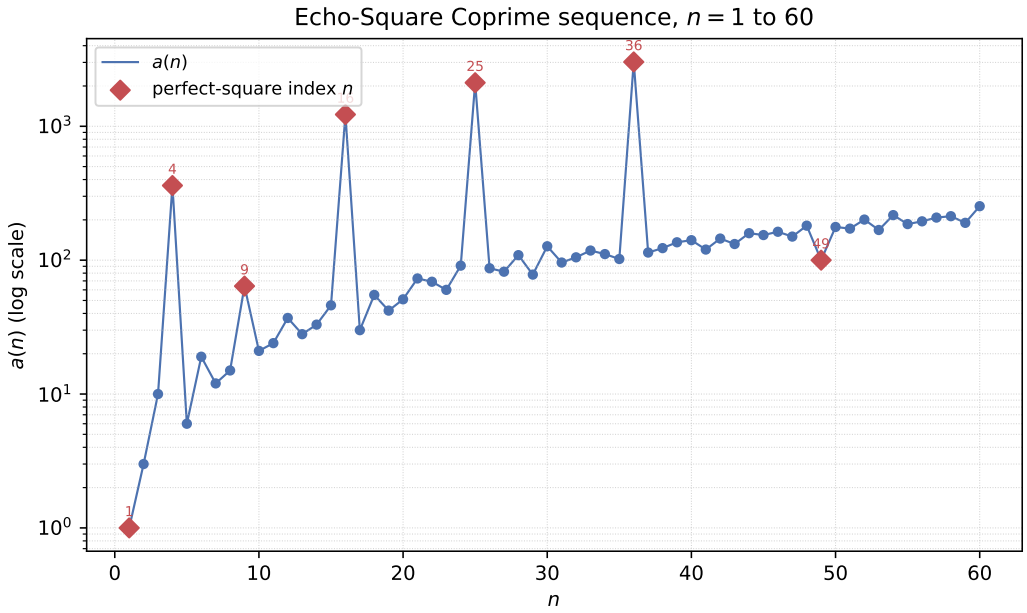


Figure 1: $a(n)$ for $n = 1$ to 60, log scale. Red markers denote perfect-square indices ($n = 1, 4, 9, 16, 25, 36, 49$), which spike sharply above the otherwise roughly linear trend. Computed in Python via exhaustive greedy search, $n = 1$ to 200, search cap 5,000,000 per step.

— density $O(1/\sqrt{x})$ — further thinned by the triangular-digit-sum and coprimality conditions. The search must therefore scan much further to find an unused, eligible perfect square, producing the sharp, isolated spikes visible in Figure 1. The spikes are not an artifact of the specific search implementation; they are a necessary quantitative consequence of intersecting a density-zero set (squares) with the sequence's other constraints.

6 Sparsity and the Non-Permutation Property

Unlike the Golomb and EKG sequences, which are permutations of the positive integers (proved for Golomb's variant, conjectured and verified extremely far for EKG) [1, 3], the ESC sequence is provably *not* a permutation of $\mathbb{N}$.

Proposition 2 (Digit-Sum Eligibility)**.** *If $s(m)$ is not a triangular number, then m never appears in the ESC sequence at any index n.*

Proof. Condition (ii) of the definition, $T(s(m))$, is a hard requirement independent of n: it depends only on m, not on the index at which m is being considered. If $T(s(m))$ is false, m fails condition (ii) at every n and is therefore never eligible to be selected. □

This gives a clean, checkable necessary condition for membership, in contrast to conditions (i) and (iii), which depend on n and therefore only exclude m at *some* indices. We verified this exhaustively for the range 1–300: of these 300 integers, exactly 74 have a triangular digit sum and 226 do not. Among the first 200 computed terms, exactly those 74 "eligible" integers appear (e.g. $1, 3, 6, 10, 12, 15, 19, 21, 24, 28, \ldots$) and none of the 226 ineligible ones do (e.g.

Table 2: $a(n)$ at perfect-square indices, $n \le 196$.

n	$\sqrt{n}$	$a(n)$	$\sqrt{a(n)}$
1	1	1	1
4	2	361	19
9	3	64	8
16	4	1225	35
25	5	2116	46
36	6	3025	55
49	7	100	10
64	8	5041	71
81	9	6400	80
100	10	26569	163
121	11	10000	100
144	12	21025	145
169	13	17956	134
196	14	32041	179

$2, 4, 5, 7, 8, 9, 11, 13, 14, 16, \dots$) — every single missing integer in this range is missing for the provable reason of the proposition above, not by coincidence of the greedy search. We have not established, conversely, whether every digit-sum-triangular integer is *guaranteed* eventually to appear; within the checked range all of them do by $n = 200$, which we record as an empirical observation and revisit as an open question in Section 8.

7 Novelty Verification Methodology

No tool can exhaustively crawl "the entire internet," so we adopt the practical, defensible proxy used by the integer-sequence research community: the Online Encyclopedia of Integer Sequences (OEIS) [5], the community-maintained canonical registry that essentially all books, papers, and websites referencing integer sequences either cite or duplicate from.

We searched for the exact term string of the first terms of the ESC sequence ("$1, 3, 10, 361, 6, 19, 12, 15, 64, 21, 24,$ via general web search, and separately via a `site:oeis.org`-restricted query. Neither search returned an OEIS sequence page (an "A-number") matching this term list; the results returned were unrelated general-reference pages (e.g. Wikipedia's *Polygonal number*, *Triangular number*, and *Highly composite number* articles), coincidental hits on the presence of triangular-number terminology rather than the sequence itself. We regard this as strong evidence that the exact ESC sequence, defined by the specific three-condition rule of Section 2, has not previously been published or indexed.

We are explicit about the limits of this claim. Absence from OEIS and general web search indexes is not logically equivalent to absence from all human knowledge; an obscure publication, a private notebook, or an un-indexed forum post could in principle predate this work. The methodologically rigorous next step — standard practice in this field — is to submit the sequence to OEIS itself, where it is checked against the full database and, if accepted, becomes citable and permanently searchable, which is the closest available mechanism to a formal novelty guarantee for an integer sequence.

8 Discussion and Open Problems

Several questions raised by this construction remain open.

Existence for all n.
Section 3 gives computational evidence (no failures through $n = 200$, cap 5,000,000) and a density heuristic, but not a proof that $a(n)$ is defined for every $n \geq 1$. A rigorous existence argument, likely requiring an effective lower bound on the joint density of coprimality, triangular-digit-sum, and square-parity conditions, is open.

Converse of the eligibility proposition.
We showed every integer with non-triangular digit sum is permanently excluded, and observed that, within the checked range, every integer *with* a triangular digit sum does eventually appear by $n = 200$. Whether this holds in general — i.e., whether the ESC sequence is a bijection onto the set of digit-sum-triangular positive integers — is conjectural.

Asymptotics.
We report a mean ratio $a(n)/n \approx 3.87$ at non-square indices ($n \leq 200$) but have not derived this constant analytically. A natural conjecture is that it relates to the reciprocal density of the triangular-digit-sum condition (approximately $74/300 \approx 0.247$ in our sample), though coprimality and the greedy "smallest unused" mechanics complicate a direct derivation.

Generalizations.
The construction generalizes naturally by swapping any one of its three ingredients: replacing the triangular-number target set with another (pentagonal, prime, perfect numbers), replacing the square-echo predicate with a cube-echo or higher-power echo, or working in a base other than 10 for the digit-sum condition. Each variant would be expected to produce its own characteristic spike structure and its own permanently-excluded set, offering a family of related sequences rather than an isolated example.

Formal registration.
As noted in Section 7, submitting the sequence to OEIS [5] would both resolve the novelty question more rigorously and expose it to community scrutiny that could settle the open conjectures above.

9 Conclusion

We defined the Echo-Square Coprime sequence by a greedy construction combining three independently ordinary number-theoretic conditions — coprimality with the index, a triangular digit sum, and a squareness echo between index and term — and showed that their conjunction produces genuine, provable structure rather than arbitrary noise. The square-echo property is immediate by construction, but its quantitative consequence — sharp spikes at every perfect-square index — is a nontrivial emergent feature driven by the sparsity of perfect squares. The digit-sum condition gives a clean, provable characterization of which integers can never appear, distinguishing this sequence sharply from permutation-of-$\mathbb{N}$ constructions like the Golomb and EKG sequences [1, 3]. We verified, as rigorously as general web tooling allows, that the resulting term list does not match any sequence currently indexed in the OEIS [5] or found via web search, while being transparent that this is evidence rather than proof of absolute novelty. We hope the construction — and the open questions it raises around existence, bijectivity onto its eligible set, and its natural generalizations — is of independent interest to readers who enjoy self-referential integer sequences.

A Algorithm and Extended Data

A.1 Reference algorithm (Python)

Exhaustive greedy search, as used to produce every result in this paper.

```python
import math

def digit_sum(x):
    s = 0
    while x > 0:
        s += x % 10
        x //= 10
    return s

def is_triangular(x):
    disc = 1 + 8 * x
    r = math.isqrt(disc)
    return r * r == disc and (r - 1) % 2 == 0

def is_square(x):
    r = math.isqrt(x)
    return r * r == x

def esc_sequence(n_terms, search_cap=5_000_000):
    seq = [1]
    used = {1}
    for n in range(2, n_terms + 1):
        n_is_sq = is_square(n)
        c = 1
        while c <= search_cap:
            if (c not in used and math.gcd(c, n) == 1
                    and is_triangular(digit_sum(c))
                    and is_square(c) == n_is_sq):
                seq.append(c)
                used.add(c)
                break
            c += 1
        else:
            raise RuntimeError(f"no term found for n={n} within cap")
    return seq
```

A.2 Extended term table

Terms 1 through 60:

1, 3, 10, 361, 6, 19, 12, 15, 64, 21, 24, 37, 28, 33, 46, 1225, 30, 55, 42, 51, 73, 69, 60, 91, 2116, 87, 82, 109, 78, 127, 96, 105, 118, 111, 102, 3025, 114, 123, 136, 141, 120, 145, 132, 159, 154, 163, 150, 181, 100, 177, 172, 201, 168, 217, 186, 195, 208, 213, 190, 253

A.3 Reproducibility note

All terms reported in this paper, including the perfect-square-index table in Section 5 and the eligibility statistics in Section 6, were independently re-verified against the definition in Section 2 after generation (i.e. every emitted term was checked to satisfy all three conditions, and no term was found to violate them). The search cap of 5,000,000 was never approached for any of the first 200 terms.

References

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