When an Infinite Hotel Runs Out of Rooms

Cardinality, Infinite Capacity, and the Limits of Hilbert's Hotel

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Abstract. The infinite hotel is a useful setting for examining the counterintuitive properties of infinite sets. A hotel with countably infinitely many rooms can accommodate additional guests even when every room is occupied, because a countably infinite set can be placed in bijection with certain proper subsets of itself. This phenomenon does not, however, imply unlimited cardinal capacity. This paper formally distinguishes *accommodation* from *room occupancy*, and separates the rules of the hotel model from the underlying set theory. Using injections, surjections, bijections, and cardinal comparison, we show that a countably infinite hotel has capacity $\aleph_0$ and can accommodate any guest set of cardinality at most $\aleph_0$ under the one-guest-per-room rule. By contrast, an uncountable guest set, such as one indexed by the real numbers, has cardinality $2^{\aleph_0}$, which Cantor's theorem shows to be strictly greater than $\aleph_0$. Hence no injection from the guests into the rooms exists, and complete accommodation is impossible. The result shows that an infinite hotel can run out of rooms not because it has a final room, but because the arriving population has strictly greater cardinality than the room set.

Keywords: cardinality; infinite sets; countable infinity; uncountable infinity; Hilbert's Hotel; $\aleph_0$; Cantor's theorem; injection; surjection; bijection

1 Introduction

The claim that an infinite hotel can never run out of rooms is a useful but incomplete piece of intuition. A hotel with rooms $r_1, r_2, r_3, \ldots$ has no last room, and yet its set of rooms can still have strictly smaller cardinality than some other infinite set. The relevant distinction is therefore not simply finite versus infinite, but a distinction *between different cardinalities of infinite sets.*

The classical Hilbert's Hotel shows that a countably infinite hotel can accommodate one new guest, and even countably infinitely many new guests, while every room remains occupied throughout. The purpose of this paper is to make precise where that phenomenon ends: we prove that if the guest set has strictly greater cardinality than the room set, then no assignment of one guest per room can accommodate every guest.

The argument rests on separating three notions that are easy to conflate. *Accommodation* — giving every guest a distinct room — is represented by an injection from guests to rooms. *Full occupancy* — leaving no room vacant — is represented by a surjection. A bijection represents both conditions holding simultaneously. Throughout, the rule that each room holds at most one guest is a *modeling assumption* about the hotel, not a theorem of set theory; set theory only decides, once that assumption is fixed, whether the required injection exists.

2 Injections, Surjections, and Bijections

We begin with the three types of function needed to compare the sizes of infinite sets.

Definition 1 (Injection)**.** A function $f : A \to B$ is *injective* if $f(a_1) = f(a_2)$ implies $a_1 = a_2$.

Distinct elements of A therefore have distinct images in B. We write $f : A \hookrightarrow B$ to mean that f is injective.

Definition 2 (Surjection)**.** A function $f : A \to B$ is *surjective* if every element of B is the image of at least one element of A. We write $f : A \twoheadrightarrow B$ to mean that f is surjective.

Definition 3 (Bijection)**.** A function is *bijective* if it is both injective and surjective. A bijection pairs the elements of two sets exactly, and so establishes an equality of size between them. We write $f : A \leftrightarrow B$ to mean that f is bijective.

3 Formal Definition of Cardinality

Let A and B be sets.

Definition 4 (Cardinal equality and comparison)**.**

$$|A| = |B| \iff \text{there exists a bijection } A \leftrightarrow B,$$
$$|A| \leq |B| \iff \text{there exists an injection } A \hookrightarrow B,$$
$$|A| < |B| \iff |A| \leq |B| \text{ and } |A| \neq |B|.$$

The cardinality of the natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$ is denoted $\aleph_0$, and any set of cardinality $\aleph_0$ is called *countably infinite*: $|\mathbb{N}| = \aleph_0$.

Remark 5. For arbitrary sets, deciding whether $|A| \leq |B|$ or $|B| \leq |A|$ (cardinal *comparability*) in general relies on the axiom of choice. This subtlety does not affect the results below: every set compared in this paper is either finite, has cardinality $\aleph_0$, or has cardinality $2^{\aleph_0}$, and these particular cardinals are directly comparable by construction, via the natural numbers, the reals, and Cantor's theorem (Theorem 15).

4 The Hotel Model: Guests, Rooms, and Rules

4.1 The set of guests

Let G denote the set of guests under consideration, with an individual guest written $g \in G$; $|G|$ is the cardinality of the guest population.

For a finite population, $G = \{g_1, \dots, g_n\}$; for a countably infinite population, $G = \{g_1, g_2, g_3, \dots\}$. For an *uncountable* guest population indexed by the real numbers, define

$$G = \{\, g_x : x \in \mathbb{R} \,\},$$

where g_x denotes the unique guest associated with the real-number index x. The map $\varphi : \mathbb{R} \to G$ given by $\varphi(x) = g_x$ is a bijection, so $|G| = |\mathbb{R}| = 2^{\aleph_0}$. The real numbers here serve only as an indexing set; the guests themselves are the elements g_x of G.

4.2 The set of rooms

Let R denote the set of rooms, with an individual room $r \in R$. For a countably infinite hotel,

$$R = \{r_1, r_2, r_3, \dots\}, \qquad |R| = \aleph_0.$$

There is no final room: for every $n \in \mathbb{N}$, the room r_{n+1} exists.

4.3 Hotel rules

The following are modeling assumptions about the hotel, not results of set theory; they specify what *kind* of function an assignment must be.

Rule 6 (Total assignment)**.** Every guest must be assigned a room: an assignment is a function $f : G \to R$.

Rule 7 (No sharing)**.** At most one guest may occupy each room. Distinct guests therefore receive distinct rooms, so f must be injective: $f : G \hookrightarrow R$.

Rule 8 (Occupancy)**.** A room counts as occupied once at least one guest has been assigned to it. Full occupancy of every room is therefore equivalent to f being surjective: $f : G \twoheadrightarrow R$.

Section 5 makes the consequences of these three rules precise.

5 Accommodation versus Room Occupancy

Accommodation and full occupancy are different conditions. Accommodation asks whether every guest can be given a distinct room; full occupancy asks whether every room has at least one guest. Rules 6–8 translate each question directly into a statement about functions, and hence, via Definition 4, into a statement about cardinality.

Proposition 9 (Accommodation)**.** *Under the one-guest-per-room rule, every guest can be accommodated if and only if there exists an injection $G \hookrightarrow R$, that is,*

$$\textit{accommodation possible} \iff \exists\, f : G \hookrightarrow R \iff |G| \leq |R|.$$

Proposition 10 (Full occupancy)**.** *Every room is occupied if and only if there exists a surjection $G \twoheadrightarrow R$:*

$$\textit{full occupancy} \iff \exists\, f : G \twoheadrightarrow R.$$

Surjectivity alone says nothing about whether rooms are shared; that is governed separately by Rule 7.

If both conditions hold for the same assignment f, then f is a bijection:

$$\text{no sharing and full occupancy} \iff f : G \leftrightarrow R.$$

In short, injections describe accommodation without sharing, surjections describe coverage of the room set, and bijections describe both simultaneously.

6 Finite Sets as a Baseline

For finite sets, cardinality coincides with ordinary counting. If there are g guests and r rooms, with at most one guest per room, then

$$g < r \Rightarrow \text{all guests accommodated, some rooms vacant;}$$

$$g = r \Rightarrow \text{all guests accommodated, every room occupied;}$$

$$g > r \Rightarrow \text{not all guests can be accommodated.}$$

Equivalently, for finite sets an injection $G \hookrightarrow R$ exists exactly when $|G| \leq |R|$, a bijection exists exactly when $|G| = |R|$, and a surjection $G \twoheadrightarrow R$ exists exactly when $|G| \geq |R|$.

This finite case is a useful baseline, but infinite sets behave differently: an infinite set can share its cardinality with a proper subset of itself, which is precisely what makes Hilbert's Hotel possible.

7 Hilbert's Hotel: One Additional Guest

Proposition 11. $\aleph_0 + 1 = \aleph_0$.

Proof. Suppose every room of the countably infinite hotel is occupied and one new guest arrives. Define f on the existing guests and the newcomer g^* by

$$r_n \mapsto r_{n+1} \quad (n \geq 1), \qquad g^* \mapsto r_1.$$

This f is injective, since $n \mapsto n+1$ is injective on $\mathbb{N}$ and r_1 is not among its outputs; it is surjective, since every room $r_1, r_2, \ldots$ appears as an image. Hence f is a bijection between the $\aleph_0 + 1$ guests and the $\aleph_0$ rooms, so no guest is turned away and no contradiction arises. □

8 Hilbert's Hotel: Countably Many Additional Guests

Proposition 12. $\aleph_0 + \aleph_0 = \aleph_0$.

Proof. Suppose countably many new guests $G_1, G_2, G_3, \ldots$ arrive. Move the existing occupant of room r_n into room r_{2n}, and seat the newcomer G_n in room r_{2n-1}:

$$r_n \mapsto r_{2n}, \qquad G_n \mapsto r_{2n-1}.$$

The even-numbered rooms receive exactly the original guests, and the odd-numbered rooms receive exactly the newcomers, so this assignment is a bijection between the combined guest set and R. A countably infinite hotel can therefore accommodate a countably infinite number of additional guests. □

9 Strict Inequality and Running Out of Rooms

Theorem 13. *If $|G| > |R|$, then no injection $G \hookrightarrow R$ exists, and consequently no complete one-guest-per-room accommodation of G is possible.*

Proof. Suppose, for contradiction, that an injection $f : G \hookrightarrow R$ existed. By Definition 4 this would give $|G| \leq |R|$, contradicting the hypothesis $|G| > |R|$. Hence no such injection exists. By Proposition 9, accommodation is possible if and only if such an injection exists; since it does not, complete accommodation is impossible. □

This is stronger than saying that one particular assignment fails: it says that *every* possible assignment must fail, because the failure is a fact about the cardinalities of G and R, not about any specific attempted pairing.

For a countably infinite hotel, $|R| = \aleph_0$. If the guest population has cardinality $2^{\aleph_0}$, then Cantor's theorem (Theorem 15 below) gives $2^{\aleph_0} > \aleph_0$, so Theorem 13 applies directly.

10 The Uncountable Guest Population

Corollary 14. *Let $G = \{g_x : x \in \mathbb{R}\}$ with $\varphi(x) = g_x$ a bijection, and let $R = \{r_1, r_2, \dots\}$ be the countably infinite room set. Then no injection $G \hookrightarrow R$ exists, and complete accommodation of G is impossible.*

Proof. Since $\varphi : \mathbb{R} \to G$ is a bijection, $|G| = |\mathbb{R}| = 2^{\aleph_0}$. Since $|R| = \aleph_0$ and $2^{\aleph_0} > \aleph_0$ by Cantor's theorem, we have $|G| > |R|$, and the result follows from Theorem 13. □

The hotel has run out of rooms in the *cardinal* sense. This does not mean a final room has been reached — there is no final room — but rather that the guest population is too large to be injected into the room population at all.

11 Cantor's Theorem

Theorem 15 (Cantor)**.** *For every set A, the power set $\mathcal{P}(A)$ (the set of all subsets of A) satisfies $|\mathcal{P}(A)| > |A|$.*

Proof sketch. The map $a \mapsto \{a\}$ is an injection $A \hookrightarrow \mathcal{P}(A)$, so $|A| \le |\mathcal{P}(A)|$. To see the inequality is strict, suppose toward a contradiction that some $f : A \to \mathcal{P}(A)$ is surjective. Let $D = \{a \in A : a \notin f(a)\} \in \mathcal{P}(A)$. By surjectivity, $D = f(a_0)$ for some $a_0 \in A$. Then $a_0 \in D \iff a_0 \notin f(a_0) = D$, a contradiction. Hence no surjection $A \to \mathcal{P}(A)$ exists, so in particular no bijection exists, and $|A| < |\mathcal{P}(A)|$. □

Taking $A = \mathbb{N}$ gives $|\mathcal{P}(\mathbb{N})| = 2^{\aleph_0} > \aleph_0$. The real numbers have the same cardinality as $\mathcal{P}(\mathbb{N})$, so $|\mathbb{R}| = 2^{\aleph_0}$, which is exactly the strict inequality used in Corollary 14.

12 General Capacity Principle

Proposition 16. *Under the one-guest-per-room rule,*

$$\textit{all guests accommodated} \iff |G| \le |R|, \qquad \textit{not all guests accommodated} \iff |G| > |R|.$$

The equality case $|G| = |R|$ means a bijection exists, so a perfect one-to-one assignment is *possible* — it does not mean that every injection $G \to R$ must itself be surjective. For instance, $f(n) = 2n$ is an injection $\mathbb{N} \hookrightarrow \mathbb{N}$ that is not surjective, even though $|\mathbb{N}| = |\mathbb{N}|$: existence of a bijection and behavior of one particular map are different questions.

13 Discussion

The infinite hotel demonstrates that "infinite" is not a single size. The cardinal $\aleph_0$ describes the size of the natural numbers, while $2^{\aleph_0}$ is strictly larger. An infinite hotel can therefore accommodate a countably infinite population while being unable to accommodate an uncountable one, even though in both cases the hotel itself has "no last room."

The argument depends on keeping three levels separate: the hotel's rules, the set-theoretic definitions, and the cardinality consequences that follow from combining them. The rule that each room

holds at most one guest converts the question of accommodation into a question about injections; set theory then determines, for the specific cardinalities involved, whether such an injection can exist.

14 Conclusion

A countably infinite hotel has room set R with $|R| = \aleph_0$. It can accommodate one additional guest, and even countably many additional guests, because $\aleph_0 + 1 = \aleph_0$ (Proposition 11) and $\aleph_0 + \aleph_0 = \aleph_0$ (Proposition 12).

Nevertheless, the hotel does not have unlimited cardinal capacity. If the arriving guest set satisfies $|G| > |R|$, then no injection $G \hookrightarrow R$ exists, and under the one-guest-per-room rule no complete accommodation is possible (Theorem 13). For an uncountable population indexed by the real numbers, $|G| = 2^{\aleph_0} > \aleph_0 = |R|$, so the countably infinite hotel necessarily runs out of rooms (Corollary 14):

$$|G| > |R| \;\Rightarrow\; \text{no injection } G \hookrightarrow R \;\Rightarrow\; \text{complete accommodation is impossible.}$$

The precise meaning of an infinite hotel "running out of rooms" is therefore *cardinal* rather than physical: the guest set has strictly greater cardinality than the room set.

Concept	**Condition**	**Hotel interpretation**
Accommodation	$\exists$ injection $G \hookrightarrow R$; equivalently $\lvert G\rvert \leq \lvert R\rvert$	Every guest has a distinct room
Full occupancy	$\exists$ surjection $G \twoheadrightarrow R$	Every room has at least one guest
Perfect occupancy	$\exists$ bijection $G \leftrightarrow R$; equivalently $\lvert G\rvert = \lvert R\rvert$	No sharing, and every room occupied
Rooms run out	$\lvert G\rvert > \lvert R\rvert$, hence no injection $G \hookrightarrow R$	Not every guest can be accommodated

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